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Research memorandum

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INVESTMENT/FINANCIAL PLANNING WITH ENDOGENEOUS LIFETIMES: A HEURISTIC APPROACH TO MIXED-INTEGER PROGRAMMING

by

Wim G. van Hulst and Jan Th. van Lieshout[1)]

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1) Both at Tilburg Catholic University.

SUMMARY

This paper deals with the question of how to achieve the best possible solution to a given investment/financial planning model of the mixed-integer linear program type when the available computer has insufficient core to allow a straightforward solution. However the paper is not concerned with inventing new arithmetical gadgets or altering the (standard) computer program. The aim is to show by example how a satisfactory result can be achieved by looking critically at the given economic problem itself: are the constraints imposed really as necessary as the textbooks claim them to be, given the economic requirements? What exactly will happen if some of these constraints are weakened to some extent? The answers to these questions concerning the given problem and a heuristic procedure led eventually to a result which is surely satisfactory from an economic viewpoint. This demonstrates once again the interaction (that should always be present) between model building and model solving.

1. INTRODUCTION.

This paper is the outcome of a convergence of two different areas of interest. In 1973 Van Hulst published a model [1] with which to establish optimal investment and financial planning, optimal in the framework of a given planning period and without regard to risk and uncertainty. The questions to be answered with the aid of this model were: which of a given set of proposed investment projects should be carried out c.q. discarded at what times and in what numbers, and which of a given set of financial funds should be raised at what times and to what amounts? This model was a mixed integer linear programming model: the project variables had to be integer, the financial variables were continuous. A fairly small and simplified numerical example however, made it clear that solving the problem thus formulated could be nothing but wishful thinking, in view of the capacity of the computer we then had at our disposal; indeed it required some ingenuity to solve the problem even without the integrality constraints. All in all a rather unsatisfactory result; so when Koks [2] remarked that 'arithmetical problems (as the required integrality of a number of variables in a real solution) when applying this and similar models to industrial practice are not solved yet', we decided to do something about it.
The second author, Van Lieshout, though not uninterested in investment problems (see [3]), was however mainly concerned with a different question. In 1974 a standard program (named XDLA mark 3) became available for solving both continuous, and mixed-integer l.p. models in the ICL 1903-A computer (64 K words core from 1974 on) of our university. This raised the question as to how integer programming is carried out by this program and just what size of problems can be solved. The user's manual did not give adequate information about this. What was more obvious than confronting the above mentioned numerical example with the mixed-integer routine? This paper is a report of the vicissitudes of this confrontation. First

we give the symbolic model in its original form, then the numerical data and the solution without regard to the integrality constraints. The value of the objective function thus determined may be seen as an ideal optimum: as soon as integrality requirements are imposed, the maximum will always be smaller. But it would be unrealistic to require all project variables to be integer, given the size of the problem: there are no less than 196 variables that ought to meet this requirement. Therefore we had to make concessions. Which we did in such a way as to make the eventual solution certainly acceptable from business point of view. Consequently the modifications we had to make to the problem were based exclusively on economic considerations and were realized by looking critically at the results (unsatisfactory as yet) obtained after each step in the procedure.
It was this procedure too, which inspired us to publish the results. Our aim was not to invent new arithmetical gadgets nor to alter the computer program, but to show by example how the user of a standard computer program that is erratic with respect to a given problem, can obtain satisfactory results by approaching the real problem in a heuristic way. We also hoped to demonstrate the interaction between model building and model solving.

2. THE MODEL.

In this section we give the investment/financial planning model in its original form. The considerations that led to this formulation are not mentioned here. As far as these are not self-evident we may refer to [1] .
A planning period is given, bounded by the points of time 0 and N. Only at the discrete points of time 0,1,2,...,N-1 can decisions be made about installing, retaining or discarding machines[2] and about raising financial funds. It

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[2] The word 'machines' is used here as an equivalent of 'durable equipment'.

is assumed that financial resources consist only of credits that have given patterns of amortisation and interest payments. For the sake of simplicity we also assume that the economic properties of machines as well as of credits are mutually independent. Finally we assume a tax rate of zero. The chosen objective in this model is to maximize the present value of the generated cash flows.

Definitions:

t_0 : the point of time at which the oldest machine still present at time 0 had been installed ($t_0 \leq 0$);

k : the number of machine types;

$x_{j\tau t}$: the number of machines of type $j(j = 1,\ldots,k)$ installed at time $\tau(\tau = t_0,\ldots,t)$ that are still present immediately after the decision taken at time $t(t = 0,\ldots,N)$;

$Q_j(\tau,t)$: the cash flow before subtracting initial investment outlays and before adding salvage values if any, generated by one machine of type j that is installed at time τ, which cash flow is measured at time t. Further to be denoted as operating surplus;

$C_j(t)$: the initial investment outlay for one machine of type j at time t;

$S_j(\tau,t)$: the salvage value at time t of one machine of type j installed at time τ;

ℓ : the number of financial resources[3]);

θ_s : the number of planning intervals[4]) at which a credit from source $s(s = 1,\ldots,\ell)$ generates interest and amortization payments;

y_{st} : the amount of credit s acquired at time $t(t = -\theta_s,\ldots,N-1)$;

d_{sth} : the interest and amortization payment per dollar of credit s that is acquired at time t which payment is due at time h;

i : the interest rate;

b_t : the amount made available at time t from elsewhere (e.g. subsidies from other sections, dividend payments, etcetera. This amount is assumed to be given and may be negative).

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3) The p^{th} interest and amortization payment of a credit acquired at time t is due at time t + p.

4) The planning period consists of N equally long planning intervals.

Objective function:

$$\text{Maximize } K = \sum_{t=0}^{N-1} e^{-it}\left[\sum_{j=1}^{k}\left\{\sum_{\tau=t_0}^{t} x_{j\tau t}\int_0^1 e^{-iw}Q_j(\tau,t+w)\,dw + \right.\right.$$

$$\left. - x_{jtt}C_j(t) + \sum_{\tau=t_0}^{t-1}(x_{j\tau,t-1}-x_{j\tau t})S_j(\tau,t)\right\} +$$

$$\left. + \sum_{s=1}^{\ell} y_{st}\left(1-\sum_{h=t+1}^{t+\theta_s} e^{-i(h-t)}d_{sth}\right)\right] + e^{-iN}\sum_{j=1}^{k}\sum_{\tau=t_0}^{N-1} x_{j\tau N}S_j(\tau,N) \quad (1)$$

Liquidity constraints:

If

$$L_0 = b_0 + \sum_{j=1}^{k}\left\{-x_{j00}C_j(0) + \sum_{\tau=t_0}^{-1}(x_{j\tau,-1}-x_{j\tau 0})S_j(\tau,0)\right\} +$$

$$+ \sum_{s=1}^{\ell}\left(y_{so} - \sum_{\tau=-\theta_s}^{-1} y_{s\tau}d_{s\tau 0}\right)$$

and

$$L_t = b_t + \sum_{j=1}^{k}\left\{\sum_{\tau=t_0}^{t-1} x_{j\tau,t-1}\int_0^1 e^{(1-w)i}Q_j(\tau,t-1+w)\,dw - x_{jtt}C_j(t) + \right.$$

$$\left. + \sum_{\tau=t_0}^{t-1}(x_{j\tau,t-1}-x_{j\tau t})S_j(\tau,t)\right\} + \sum_{s=1}^{\ell}\left(y_{st} - \sum_{\tau=-\theta_s+t}^{t-1} y_{st}d_{s\tau t}\right),$$

$$(t = 1,\ldots,N),$$

then the liquidity constraints can be written shortly as:

$$\sum_{z=0}^{t} e^{(t-z)i} L_z \geqq 0 \qquad (t = 0,\dots,N) \qquad (2)$$

$$\sum_{z=0}^{t} e^{(N-z)i} L_z + \sum_{j=1}^{k} \sum_{\tau=t_0}^{N-1} x_{j\tau N} S_j(\tau,N) +$$

$$- \sum_{s=1}^{\ell} \sum_{\tau=N-\theta_s+1}^{N-1} \sum_{t=N+1}^{\tau+\theta_s} e^{-(t-N)i} y_{s\tau} d_{s\tau t} \geqq 0 \qquad (3)$$

Credit constraints:

$$y_{st} \leqq y_{st}(\max) \qquad (s = 1,\dots,\ell;\ t = 0,\dots,N-1) \qquad (4)$$

Other constraints:

$$x_{j\tau t} - x_{j\tau,t-1} \leqq 0$$

$$(j = 1,\dots,k;\ \tau = t_0,\dots,t-1;\ t = 0,\dots,N-1) \qquad (5)$$

$$x_{j\tau N} - x_{j\tau,N-1} = 0 \qquad (j = 1,\dots,k;\ \tau = t_0,\dots,N-1) \qquad (6)$$

$$x_{j\tau t} \geqq 0 \qquad (j = 1,\dots,k;\ \tau = t_0,\dots,t;\ t = 0,\dots,N) \qquad (7)$$

$$x_{jNN} = 0 \qquad (8)$$

$$y_{st} \geqq 0 \qquad (s = 1,\dots,\ell;\ t = 0,\dots,N-1) \qquad (9)$$

$$y_{sNN} = 0 \tag{10}$$

$$x_{j\tau t}: \text{ integer } (j = 1,\ldots,k;\ \tau = t_0,\ldots,t;\ t = 0,\ldots,N) \tag{11}$$

3. THE DATA.

An enterprise has a planning period consisting of 10 intervals, bounded by the equidistant points of time 0 to 10. At time 0, which can be identified with 'now', two machines of type 1 and three machines of type 2 are present which had been installed at times -2 and -1 respectively. Type 1 is offered on the market of capital goods during the entire planning period, type 2 can be obtained at time 5 on the latest. Further there are two different types that may be useful for our firm, namely types 3 and 4 of which the firm does not poses any specimen as yet. Type 3 is offered during the entire planning period, type 4 from time 4 to 10. Further data about operating surplusses, salvage values and initial investment outlays are given in the following tables. All data in these tables are measured in thousands of dollars. It is assumed that operating surplusses become available at the end of each planning interval and that receipts of salvage values and expenditures of initial investment outlays fall at the start of each interval. Interest rate is 10 percent.

Operating surplus of one machine of type 1: $Q_1(\tau,t)$.

τ \ t	-1	0	1	2	3	4	5	6	7	8	9	10
-2	60	50	40	25	20	15	10	0	-5	-10	-15	-20
0			80	70	65	60	55	50	30	25	20	15
1				85	70	60	50	45	30	30	20	15
2					90	80	70	60	50	40	30	20
3						100	100	100	100	95	95	90
4							110	105	100	95	85	80
5								115	110	100	95	90
6									120	100	95	90
7										130	125	120
8											150	140
9												160

Salvage values of one machine of type 1: $S_1(\tau,t)$ and its initial investment outlay $C_1(\tau)$.

τ \ t	-1	0	1	2	3	4	5	6	7	8	9	10	$C_1(\tau)$
-2	115	100	90	70	40	20	10	0	0	0	0	0	140
0			135	120	100	90	80	70	65	60	55	50	180
1				150	135	120	90	70	45	35	30	25	200
2					200	170	160	150	130	110	95	85	240
3						230	220	220	215	215	210	210	250
4							240	220	200	170	140	100	300
5								270	265	260	255	252	320
6									280	220	170	130	350
7										320	310	280	400
8											380	350	425
9												390	450

Operating surplusses of one machine of type 2: $Q_2(\tau,t)$.

τ \ t	0	1	2	3	4	5	6	7	8	9	10
-1	65	60	55	52	45	41	38	30	10	0	-5
0		67	62	55	48	45	40	40	20	10	8
1			75	60	55	48	45	42	30	35	20
2				80	67	65	55	50	48	46	42
3					80	75	70	68	62	57	50
4						80	70	60	50	40	30
5							80	65	45	30	25

Salvage values of one machine type 2: $S_2(\tau,t)$ and its initial investment outlay: $C_2(\tau)$.

τ \ t	0	1	2	3	4	5	6	7	8	9	10	$C_2(\tau)$
-1	140	130	120	115	110	80	60	25	10	0	0	154
0		135	125	120	115	105	75	40	20	10	0	150
1			135	125	118	105	80	45	30	15	5	156
2				130	120	115	95	50	45	25	10	156
3					125	122	120	70	65	35	15	196
4						130	125	75	70	45	20	150
5							115	85	80	55	30	144

Operating surplusses of one machine of type 3: $Q_3(\tau,t)$

τ \ t	1	2	3	4	5	6	7	8	9	10
0	110	90	88	87	60	50	45	32	28	15
1		120	105	100	85	70	55	50	48	45
2			149	140	130	60	40	20	5	0
3				180	172	170	166	160	155	60
4					190	140	120	95	75	60
5						190	160	140	100	80
6							195	190	180	175
7								196	185	140
8									196	180
9										200

Salvage values of one machine of type 3: $S_3(\tau,t)$ and its initial investment outlay $C_3(\tau)$.

τ \ t	1	2	3	4	5	6	7	8	9	10	$C_3(\tau)$
0	300	240	140	105	102	100	95	90	80	60	373
1		257	110	95	90	85	80	75	70	50	342
2			180	137	100	97	95	90	60	60	365
3				480	435	433	340	330	325	310	600
4					405	390	365	360	347	320	469
5						420	409	400	390	350	484
6							499	450	440	309	580
7								500	473	380	550
8									420	400	544
9										490	628

Operating surplusses of one machine of type 4: $Q_4(\tau,t)$.

τ \ t	5	6	7	8	9	10
4	80	70	60	50	40	30
5		95	92	88	75	68
6			96	82	74	65
7				100	92	85
8					100	90
9						103

Salvage values of one machine of type 4: $S_4(\tau,t)$ and its initial investment outlay: $C_4(\tau)$.

τ \ t	5	6	7	8	9	10	$C_4(\tau)$
4	100	90	88	85	80	75	205
5		150	133	125	100	90	222
6			145	140	120	110	162
7				175	140	115	190
8					150	120	190
9						150	190

Further data:

$b_t = 0 \qquad (t = 0,\dots,N)$

$\ell = 2$

$y_{1t}(\max) = 80 \qquad (t = 0,\dots,9).$

This credit has to be repaid by a five periods 8 per cent annuity, hence

$d_{1th} = 0.25046 \qquad (h = t+1,\dots,t+5);$

$$y_{2t}(\max) = 50 \qquad (t = 0,\ldots,9).$$

This is a short term credit with a maturity term of one period; interest 12 per cent. Hence

$$d_{2t,t+1} = 1.12.$$

At time 0 there are no more payments due resulting from previous credits.
If one substitutes these data into the l.p. model, one has a problem with 216 variables and 196 constraints apart from the non-negativity and the integrality constraints. Of course the constraints (6), (8) and (10) were not taken separately but had already been substituted into the model. The constraints (4) were regarded as so-called bounds, which, as is generally known, results in s considerable reduction of computer time.

4. THE SOLUTION WITHOUT REGARD TO INTEGRALITY CONSTRAINTS.

In the first instance the problem was solved without taking into account the integrality constraints. After 126 iterations (because of the 164 constraints (5) the problem was very degenerated) the optimal solution was obtained of the problem thus reduced. The most important results were:

$$K = 10{,}409.73$$

$$x_{200} = x_{201} = x_{202} = x_{203} = 5.00$$

$$x_{211} = 2.49$$

$$x_{222} = x_{223} = x_{224} = 5.56$$

$$x_{133} = 2.94$$

$$x_{244} = x_{245} = 14.33$$

$$x_{255} = 14.73$$

$$x_{466} = 34.83$$

$$x_{477} = x_{478} = x_{479} = x_{4710} = 44.04$$

$$x_{488} = x_{489} = x_{4810} = 69.39$$

$$x_{499} = x_{4910} = 57.71$$

The remaining x variables (basic or not) are zero.

$$y_{1t} = 80 \qquad (t = 0,\ldots,9)$$

$$y_{2t} = 50 \qquad (t = 0,\ldots,9)$$

That this optimal solution shows an integer number of machines to be installed at time 0 is a pure coincidence. The solution is illustrated in figure 1.

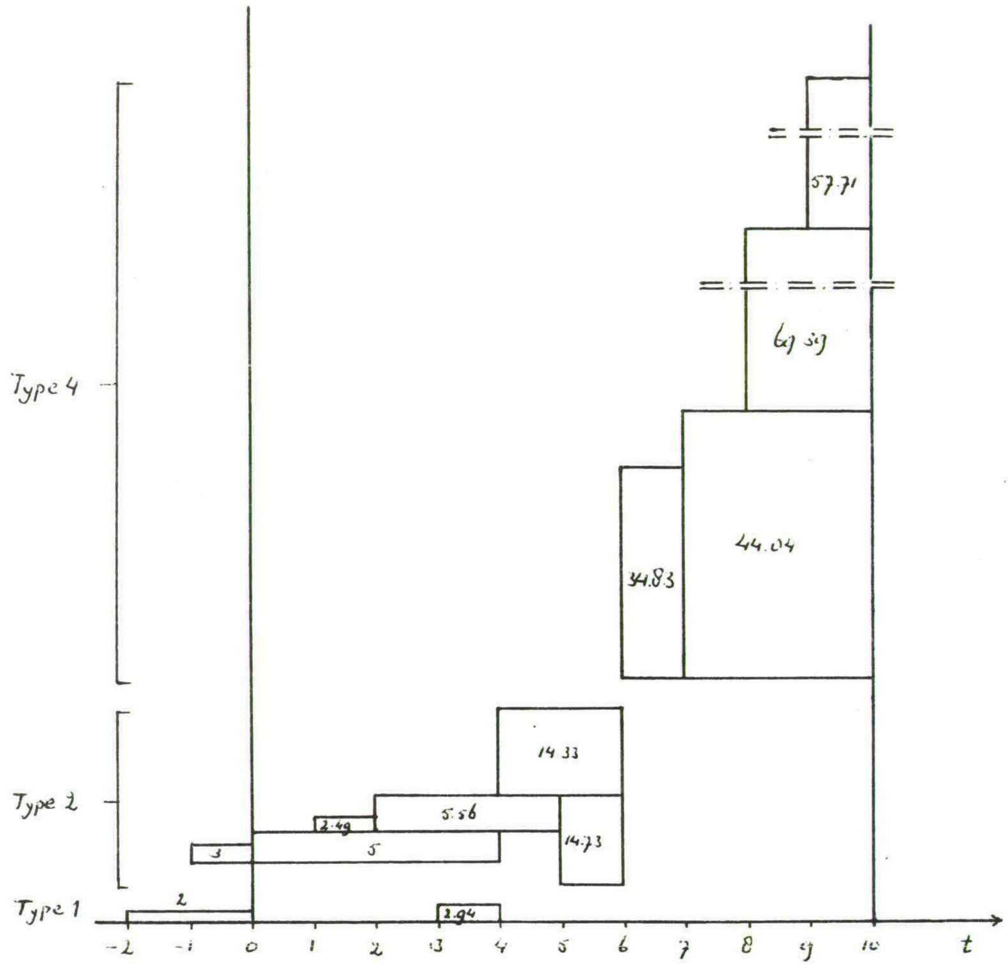

Figure 1. THE CONTINUOUS SOLUTION

5. THE WAY TO AN OPTIMAL MIXED-INTEGER SOLUTION.

Although we could guess what the outcome would be, we first tried to see if rounding off the appropriate non-integer results produced a satisfactory result. It was soon clear that this produced only infeasible solutions, so we had to look for a different way. It was immediately clear that it was impossible to impose the integrality requirement on 196 variables. So for the time being this requirement was imposed only on those x variables that concerned the points of time $t = 0$ to 3. Thus the problem was reduced to a mixed-integer l.p. problem with 38 integer variables, which seemed reasonable at first sight. As there were quite considerable differences in the levels of the continuous solution we did not set bounds to these variables a priori, except, of course, the trivial non-negativity constraints. Even so the size of the problem still proved to be too large for the available computer capacity. It is true that we found some feasible solutions but it would have taken too long to check whether one of them was optimal. We did however find an 'acceptable' solution after one hour mill time: the value of the objective function differed less than 2 per cent of this value in the continuous case.

But what problem did we actually 'solve'? Could it still be regarded as the real problem? We proceeded by subjecting this latter problem to a critical investigation. A number of good reasons might be suggested for restricting the integrality requirements to those x variables that are related to decisions to be taken at early points of time. Firstly the question is concerned in principle only with those variables that represent decisions to be taken immediately, i.e. the variables $x_{j\tau 0}$. But we are also concerned with the 'career' of the machines represented by the $x_{j\tau 0}$ variables, if only in order to determine the depreciation plan. Moreover decisions at time 0 are influenced by later decisions concerning both the choice of machine type and the level. Therefore it is not

sufficient to impose the integrality requirement only on the $x_{j\tau_0}$ variables, also the x variables related to subsequent times should meet this requirement.
Secondly the continuous solution shows some remarkable characteristics. As can be seen from figure 1, the level of the solution for $t > 5$ is much higher than for $t \leqq 5$; it might even be regarded as an absolutely high level. Further, it turns out that machines installed in the first half of the planning period are scrapped at $t = 6$ at the latest. Moreover the continuous solution is very stable with regard to the composition of the basic solution: the boundaries between which the coefficients of the objective function and the right hand sides of the constraints may move without a change in the composition of the basic solution, are generally very wide. If such a boundary is trespassed in respect of a variable or a right hand side related to $t > 5$ then the composition of the basic solution still remains unchanged for $t < 5$. All these findings led us to an acceptance of the hypothesis that preliminary decisions are not significantly influenced by the imposition or non-imposition of integrality restrictions on x variables for $t > 5$. A test (though not indeed a waterproof one) to this hypothesis might be found in the differences between the value of the objective function and the levels of the x variables for $t > 5$ in the eventual optimal mixed integer solution on the one hand and these magnitudes in the continuous solution on the other hand.
But now the border had been shifted from $t = 3$ to $t = 5$ so the number of integer variables had increased again. In order to meet this objection it was necessary to re-examine some of the assumptions on which the model had originally been based. It was at first assumed that decisions about installing and discarding machines could be made only at discrete points of time t. This assumption is fully maintained with regard to the installation of new machines. In our case this means that all x_{jtt} $(0 \leqq t \leqq 5)$ must be integer. However we abandoned the assumption concerning the scrapping of economically worn-out

machines; from now on we also allow a machine to be scrapped anywhere between two discrete points of time. How this works out in formulating integrality constraints we shall discuss in the next section, but first we want to discuss the consequences of this weaker assumption.
Scrapping a machine between times means that a variable $x_{j\tau,\tau+a}$ (a > 0) does not need to be integer. This statement may not be extended over more than one value of a without some further restrictions, but we shall come to this point later. The following is an example of the result that may be expected:

$$x_{100} = 4;\ x_{101} = 3.8;\ x_{102} = 3.$$

This must be interpreted as: at time 1 there are still 4 machines present, but at time 1 + 0.8 one of them is scrapped. In other words we consider the event '0.8 machine during one planning interval' as equivalent to the event 'one machine during 0.8 interval'. However this requires further assumptions concerning the behaviour of the coefficients of this variable in objective function and liquidity constraints. In these relations this variable appears multiplied by an operation surplus Q and a salvage value S. Firstly the equivalence of the events just mentioned requires the assumption that the operating surplus during a planning interval, discounted at the start of it, shows a linear graph. Only if this is true, then indeed

$$n_1/n_2 \int_0^1 e^{-iw}\, Q_j(\tau,t+w)dw = \int_0^{n_1/n_2} e^{-iw}\, Q_j(\tau,t+w)dw,$$

where n_1 and n_2 are natural numbers and $n_1 < n_2$.
Secondly a similar assumption needs to be accepted concerning the salvage value. The pertinent terms in the objective function in the above mentioned example are:

$$e^{-i}\{0.2\ S_1(0,1) + 0.8\ e^{-i}\ S_1(0,2)\}$$

which is equivalent to

$$e^{-1.8i}\ S_1(0,1.8)$$

if the graph of the discounted salvage value within one planning interval is linear.
However difficulties arise concerning the liquidity constraints. According to the model-formulation there is at time 1 a contribution to the liquidity to an amount of 0.2 S(0,1), plus one at time 2 to an amount of 0.8 $S_1(0,2)$. If we interpret the non-integer value of x in the way described above, these contributions do not appear at all, but between these two points of time an amount becomes avaliable that is a little larger than the weighed average of these amounts (a little larger because of the effect of discounting). For the sake of liquidity we therefore have to assume that this amount, becoming available in the meantime, can by adequate financing be spread over the two points of time.

6. REFORMULATING THE MODEL.

As restricting the integrality requirements to the first half of the planning period does not really affect the formulation of the model, and as the weakening of these requirements concerning the scrapping time does, we shall first consider the latter. In section 7, at the solving phase, both matters will be included.
In the previous section it was stated that not all non-integer results of $x_{j\tau,\tau+a}$ $(a > 0)$ are feasible. A set of such results ought to make it possible to scrap at least one machine between times, without coming into conflict with the constraints (5). It can easily be shown that there are two feasible ways in which a x variable not related to a new investment (so $t > \tau$) may have a non-integer value:

a. it is immediately preceded by an integer result in the sequence $x_{j\tau,\tau+a}$ for given j and τ.
b. if its immediate predecessor in the sequence is not integer, then the interval bounded by the value of the variable under analysis and the value of its immediate predecessor must be wide enough to contain one natural number at least.

If one takes into account these considerations, the model of section 2 can be reformulated as:

Maximize (1)
Subject to (2) to (4), (7) and (9)
while (5) and (11) are replaced by

$$\left.\begin{array}{l} x_{j\tau t} - z_{j\tau t} \geqq 0 \\ z_{j\tau t} - x_{j\tau,t+1} \geqq 0 \\ z_{j\tau t}\text{: integer} \end{array}\right\} \begin{array}{l} (j = 1,\dots,k;\ \tau = t_0,\dots,N-1; \\ t\in[\{(0,2,4,\dots)\cup(N-1)\}\cap\{\tau+2,\dots,N-1\}] \text{ if} \\ \tau \text{ is even and} \\ t\in[\{(1,3,5,\dots)\cup(N-1)\}\cap\{\tau+2,\dots,N-1\}] \text{ if} \\ \tau \text{ is odd or negative.} \end{array} \tag{5a}$$

$$\left.\begin{array}{l} x_{jtt} - x_{jt,t+1} \geqq 0 \\ x_{jtt}\text{: integer} \end{array}\right\} (j = 1,\dots,k;\ t = 0,\dots,N-1) \tag{5b}$$

and (6), (8) and (10) are substituted into the model itself. The rather strange looking counting set of t may perhaps require some explanation. For given j the integrality requirements can be illustrated by the following scheme:

τ \ t	0	1	2	3	4	...	N-1
t_0		*		*		...	*
.		.		.			.
.		.		.			.
.		.		.			.
-1		*		*		...	*
0	*		*		*	...	*
1		*		*		...	*
2			*		*	...	*
3				*		...	*
4					*	...	*
.							.
.							.
.							.
N-1							*

The asterisks denote an integrality requirement for x_{jtt} (at the main diagonal of the lower part of the scheme) and for $z_{j\tau t}$ (the remaining part). In this way the above verbally formulated requirements are met. Note that for $\tau < 0$ the 'variable' $x_{j\tau,-1}$ is a datum and hence always integer. Note further the integrality requirement for $t = N-1$; this is necessary because $x_{j\tau,N-1} = x_{j\tau N}$ is required, so inside of the interval (N-1,N) no scrapping is allowed.
The number of integer variables in the most unfavourable case (i.e. if all possible types had been installed at t_0 and are available during the entire planning period) equals $k(\frac{1}{4}N^2+N+\frac{1}{2}N|t_0|)$ if N is even and $K\{\frac{1}{4}N^2+N+\frac{1}{2}|t_0|(N+1)-\frac{1}{4}\}$ if N is odd. In our numerical example there are, according to the reformulated model, 122 integer variables; so we succeeded in reducing the model by 74 integer variables though the model actually became mor general. However the remaining number of integers is still too large to be solved in straightforward

way. Therefore we shall now describe a stepwise procedure which, as can easily be seen, will lead to the optimum, but will require a lot of patience from the analyst if it is carried out to the end. The procedure is as follows:

a) Start by adding the constraints:

   x_{jtt}: integer $\quad (j = 1,\ldots,k;\ t = 0,\ldots,N-1)$

   to the (continuous) model (1) to (7) and (9) and solve it[5].

b) For each sequence $x_{j\tau,\tau+a}$ $(a > 0, j$ and τ given) resulting from the previous step, determine which variable first comes into conflict with the requirement:

   $$[x_{j\tau t}]^{-} \geqq x_{j\tau,t+1}$$

   For these variables, insert constraints of the form (5a). If there are no (more) variables in conflict with the above-mentioned requirement, the optimal solution has been obtained. If there still are, proceed to c).

c) Solve the model resulting from step b. Then go back to b).

This procedure makes it possible to insert only those integrality constraints that are absolutely necessary: no integrality constraints are imposed on variables that will not appear in the eventual solution with a value larger than zero. But as one does not know before how many integrality constraints are necessary, two things may happen:

1) either one has to go through a tedious and very long procedure before obtaining the optimal result;

2) or because of subsequent adding constraints (5a) the model grows beyond the capacity of the computer and the optimal

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[5] To accelerate the solving procedure it is not difficult to set appropriate upper bounds. The same can be said about the solution in step c. However if in a solution a value equals a bound, the latter might be too low. Increase it and solve again.

solution will not be obtained at all; one does, however, obtain a better approximation of it after each step, and this approximation meets integrality requirements for the earlier part of the planning period.

7. A HEURISTIC PROCEDURE AND SOLUTION.

Now we return to our numerical example. We did not carry out the procedure of section 6 to the end because we were interested, not in 'the' optimal solution, but in a satisfactory one that could be obtained in a relatively short time. We defined 'satisfactory' as when the difference between the optimal value of the objective function in the continuous model and this value in the eventual mixed-integer solution is less than 0.5 per cent. By 'relatively short time' we meant that the satisfactory solution should be obtained in a time shorter than five times the time needed for solving the continuous model.

Firstly we again took our hypothesis of section 5 concerning the restriction of integrality requirements to $t = 0$ to $t = 5$. This hypothesis can easily be inserted into the procedure of section 6 by replacing at step a) N-1 by T where T denotes the last point of time than an integrality requirement should be met.

Secondly, though we could already see that a continuous solution would not do, this solution does give a considerable amount of information. The most important is that one can immediately see which investment projects will certainly not be realized because of their huge shadow prices. So by removing these variables the model can be reduced. In our numerical example this was the case with respect to e.g. all x variables corresponding to type 3.

Now we went through the following procedure:

a) Solve the continuous model (1) to (7) and (9) and remove 'superfluous' variables.

b) Put a (new) lower bound to the objective function (in our case: 99.5 percent of the value of the objective function in the continuous model).

c) For each sequence $x_{j\tau,\tau+a}$ ($a > 0$, j and τ given) in the solution found up to now, determine which variable first comes into conflict with

x_{jtt}: integer ($t = 0,\ldots,T = 5$)

and

$[x_{j\tau t}]^{-} \geqq x_{j\tau,t+1}$ ($t \leqq T = 5$)

For these variables insert constraints (5a) and (5b) into the model. If there are no (more) such variables, proceed to e), or else to d).

d) Solve the reformulated model. If there is no solution because the cut off percentage of step b) is too tight, loosen it and solve again if the new lower bound can still be considered as satisfactory. Go back to c).

e) Analyze shadow prices and penalties and decide whether a tightening of the cut - off percentage of step b) would result in a considerable improvement. If it does, go back to step b), or else stop.
Of course: watch the clock!

In this way we found the required satisfactory solution in one step after solution of the continuous problem; this step took 30 minutes mill time. The most important results are:

$K = 10,359.00$

$x_{200} = 5$	$x_{245} = 13.93$
$x_{201} = 4.41$	$x_{255} = 15$
$x_{202} = 3.18$	$x_{466} = 34.66$
$x_{203} = 3$	$x_{477} = x_{478} = x_{479} = x_{4710} = 43.83$
$x_{211} = 3$	$x_{488} = x_{489} = x_{4810} = 69.06$
$x_{222} = x_{223} = 7$	$x_{499} = x_{4910} = 57.43$
$x_{224} = 5.87$	$y_{1t} = 80 \quad (t = 0,\ldots,9)$
$x_{133} = 3$	$y_{2t} = 50 \quad (t = 0,\ldots,2,4,\ldots,9)$
$x_{244} = 14$	$y_{23} = 29.68$

From further results it was clear that the maximum present value might be obtained was less than 10,380.00. To verify whether this solution was optimal would have taken at least 90 minutes mill time, so this solution was considered as satisfactory. At step d) the model contained 207 non-negative variables of which 11 integer ones and 186 constraints.
This solution is pictured in figure 2. Comparison of it to the continuous solution (see also figure 1) shows that the present value is indeed less than 0.5 percent lower. The composition of the investment program is the same and there are no great differences between the levels. The differences between the levels for $t > 5$ can be all but neglected, so our hypothesis that investments after $t = 5$ hardly influence the strategy in the first half of the planning period is supported by these results.

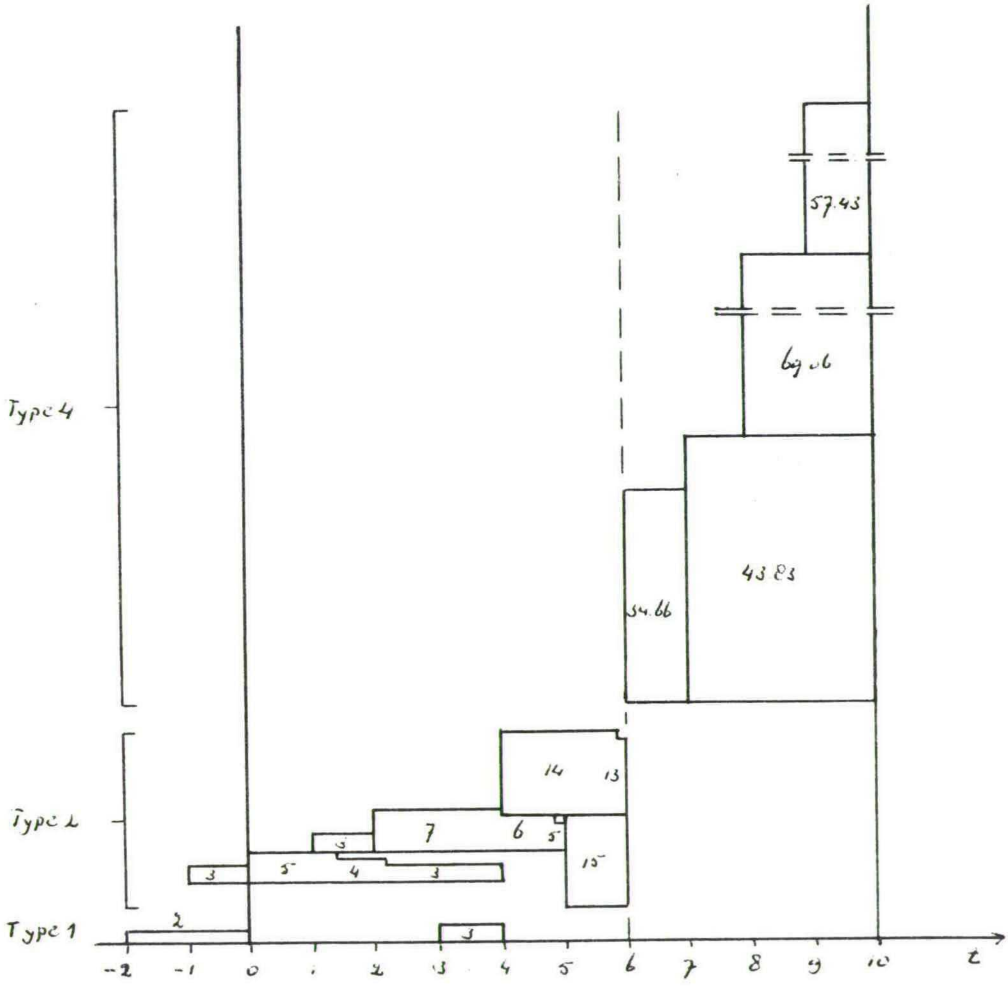

Figure 2. THE EVENTUAL MIXED-INTEGER SOLUTION

9. FINAL REMARKS.

A point deserving attention is that at each step in the heuristic procedure the entire problem must be solved right from the beginning. The procedure would run much faster if at each following step one could start from the level where the previous step became stuck: at any rate this was a solution somewhere down the tree and there is no reason to repeat the cutting process from the beginning. However this is what does happen because adding new integrality requirements and altering bounds must take place outside the proper arithmetic process. In the meantime results already found are removed from the memory; the standard program does not offer the possibility of storing intermediate results in a form accessible to the computer. We think that changing the program in this sense should certainly be recommended.

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